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Computational Linear and Commutative Algebra

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This book is an lovely blend of commutative and linear algebra. The authors previously collaborated on the two-volume work *Computational Commutative Algebra*, published by Springer in 2000 and 2005. Their new book has a different structure (no exercises or tutorials) but fortunately preserves the irreverent humor of the earlier volumes. It is an interesting and original text. The motivation for the book comes from the study of systems of polynomial equations

$$(1) \qquad f_1(x_1,\ldots,x_n) = \cdots = f_m(x_1,\ldots,x_n) = 0$$

that have finitely many solutions over the algebraic closure of the coefficient field K. In this situation, the quotient ring $R = K[x_1,\ldots,x_n]/\langle f_1,\ldots,f_m\rangle$ is a finite-dimensional vector space over K, and an element $f \in R$ gives the linear map $\theta_f : R \to R$ defined by $\theta_f(g) = fg$. This is where the authors quote a Woody Allen joke that involves the biblical injunction "be fruitful and multiply." Since R is commutative, the mutliplication maps θ_f form a commuting family of endomorphisms of a finite-dimensional vector space. One of the central themes of the book is that such families are indeed a fruitful object of study.

Chapter 1 considers a K-linear map $\varphi : V \to V$, where $\dim_K(V) < \infty$. Since the authors want K to be arbitrary, the eigenvalues and eigenspaces of φ are less important. Rather, the minimal polynomial of φ takes center stage, and its irreducible factors, called *eigenfactors*, assume the role of eigevalues. An eigenfactor has an eigenspace and a generalized eigenspace, and V is the direct sum of the generalized eigenspaces, as one would expect. Let me also comment on terminology. In linear algebra, a linear map $V \to V$ whose minimal and characteristic polynomials agree is *non-derogatory*. This is relevant because the existence of a non-derogatory $\theta_f : R \to R$ has strong consequences for the structure of R. The authors introduce the term *commendable* to replace non-derogatory, arguing that "Would you call a beautiful painting 'non-ugly'?" My own feeling about commendable vs. non-derogatory is that it is similar to smooth vs. non-singular in algebraic geometry. I think there is place for both.

In Chapter 2, things get more sophisticated as the focus shifts to families of commuting endomorphisms of V. Such a family $\mathscr{F}$ is a K-algebra with V as a finitely-generated $\mathscr{F}$-module. The authors note that "a considerable dose of commutative algebra" is required, but this allows them to generalize the classical fact that *commuting diagonalizable matrices can be simultaneously diagonalized*, even when no eigenvalues lie in K. The maximal ideals of the algebra $\mathscr{F}$ and their associated kernels in V are essential tools. The primary decomposition of $\{0\} \subseteq \mathscr{F}$ is also important.

Chapter 3 is devoted to special types of families, including ones that contain a commendable element, naturally called commendable families. As an example of what this means algebraically, the authors note that when $\mathscr{F}$ is a field, it is commendable if and only if it has a primitive element in the sense of field theory. Another interesting family uses the dual map $\varphi\check{} : V^* \to V^*$ of $\varphi \in \mathscr{F}$ to create that dual family $\check{\mathscr{F}}$ that features in the next chapter.

Chapter 4 is for me the heart of the book. The focus is on $R = K[x_1, \ldots, x_n]/\langle f_1, \ldots, f_m \rangle$, $\dim_K(R) < \infty$. The commuting family $\mathscr{F} = \{\theta_f \mid f \in R\}$ is naturally isomorphic to R, so Chapters 2 and 3 apply to R. For example, if $\{0\} = \bigcap_i \mathfrak{q}_i$ is the primary decomposition of $\{0\} \subset R$, then the factorization into local rings

$$R \simeq \textstyle\prod_i R/\mathfrak{q}_i \tag{2}$$

expresses R as the product of the joint generalized eigenspaces of $\mathscr{F}$. Also, the dual family $\check{\mathscr{F}}$ acts natually on the dualizing module ω_R. By Chapter 3, $\mathscr{F}$ is commendable if and only if R is Gorenstein, a splendid link between linear algebra and commutative algebra.

The decomposition (2) highlights the importance of computing primary decomposition. This is a challenging problem, and Chapter 5 presents of state-of-the-art discussion for the rings and ideals considered here.

The book culminates in Chapter 6, which is devoted to the problem of finding the solutions of the system (1). The basic idea is that if $p \in K^n$ is a solution of (1) and $f \in R$, then $f(p)$ is an eigenvalue of both θ_f and $\theta_f\check{}$. This corresponds to 1-dimensional joint eigenspaces of $\mathscr{F}$ and $\check{\mathscr{F}}$, leading to some nice solution methods. The authors also explore what happens over finite fields and extension fields.

The book is well-written and includes many examples. Each chapter begins with a summary that motivates the (often substantial) mathematics to follow, and every method is accompanied by an algorithm, justifying the word "Computational" in the title. Some parts of the book are self-contained, while others require an acquaintance with their earlier volumes. For example, Algorithm 2.14 will baffle someone not familiar with the Buchberger-Möller algorithm. The book contains many new results and concepts, along with known ideas drawn from a widely scattered literature. Here they appear in a coherent general context. Overall, this book is a worthy contribution to both linear and commutative algebra.

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